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Statistics and scaling in one-dimensional disordered systems

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Abstract. We present a new calculation of the statistical cumulants of $-\ln|t|^2$ and Θ where $t = |t| \exp(i\Theta)$ is the transmission of a one-dimensional (1D) disordered system. We find that both variables are normally distributed in the long-length limit and that in general the distributions obey a two-parameter scaling. However, it does not follow that the distributions of $|t|^2$ or $1/|t|^2$ are log–normal. We find that $|t|^2$ is never log–normal while $1/|t|^2$ is so only for weak disorder. For the 1D Anderson model we show that there is a crossover to a single-parameter scaling in the weak-disorder limit.

1. Introduction

Much of the present understanding of transport in disordered systems is based on the scaling theory of localisation [1], predicting as it does the existence of a mobility edge in three dimensions (3D) with a critical exponent of unity and that all the states in 1D and 2D systems are localised by arbitrary disorder. The theory is crucially dependent on a single-parameter scaling assumption embodied in the definition of the beta function

$$\mathrm{d}\ln g/\mathrm{d}\ln L = \beta(\ln g). \tag{1}$$

Here g is the dimensionless conductance. However, as is now widely appreciated equation (1) cannot be correct as it stands because it ignores fluctuation effects. The basic problem is that in the absence of inelastic effects the conductance of a disordered system is sensitive to the exact microscopic arrangement of the impurities. In the localised regime the fluctuations in g diverge exponentially as the system size increases, and even in the metallic regime there are the so-called universal conductance fluctuations [2].

The lack of any self-averaging of g prompted detailed studies of 1D systems, which showed [3–13] that the relevant scaling variable is $\ln(1 + R)$, R being the dimensionless resistance, and that this variable has a normal distribution in the long-length limit. Since a normal distribution is determined by two parameters, the mean and the variance, this implies a two-parameter scaling for the distribution of $\ln(1 + R)$ casting doubt on the single-parameter scaling theory unless some relation can be found between the mean and the variance of $\ln(1 + R)$. Just such a relationship has been found for the weak-disorder limit of the random-phase model [12] and for the Gaussian random potential model [5, 11]

$$\mathrm{var}[\ln(1 + R)] = 2\langle\ln(1 + R)\rangle. \tag{2}$$

As we shall see below this result also holds under certain conditions for the Anderson model. However, in general the Anderson model obeys a genuine two-parameter scaling.

In this paper we approach the problem through a systematic calculation of the statistical cumulants of $-\ln|t|^2$ where t is the transmission coefficient of a 1D disordered system. We find that $-\ln|t|^2$ is normally distributed in the long-length limit and obeys a two-parameter scaling, reducing to a single-parameter scaling only under certain conditions. However, the real value of our approach presents itself in the calculation of the higher cumulants which allows us to determine how accurate the normal approximation actually is. We find that the positive moments of $|t|^2$ never agree with the log–normal distribution for $|t|^2$, and the negative moments do so only for weak disorder. The implications of this result for the distributions of conductance and resistance are discussed in the conclusion. We also derive the asymptotic form of the distribution of the phase of t and find that this always obeys a two-parameter scaling.

2. Scaling of the distributions

2.1. A general expression for the cumulants

In this section we derive an expression for the statistical cumulants of $-\ln|t|^2$ and Θ where $t = |t|\exp(\mathrm{i}\Theta)$ is the transmission coefficient for waves travelling through a 1D disordered system and from this deduce the asymptotic form of the distributions of both quantities. We describe the scattering properties of the system by a transfer matrix $\mathbf{T}$ which relates incident- and reflected-wave amplitudes at the left (a_+ and a_- respectively) to those at the right (b_+ and b_-) of the system.

$$\begin{pmatrix} b_+ \\ b_- \end{pmatrix} = \mathbf{T}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}. \tag{3}$$

The transfer matrix $\mathbf{T}$ is a product of transfer matrices

$$\mathbf{T} = \prod_{n=1}^{L} \mathbf{M}_n \tag{4}$$

where $\mathbf{M}_n$ $n = 1, \ldots, L$ are independent random matrices with some probability distribution $p(\mathbf{M})$. If the system is time-reversal symmetric then $\mathbf{T}$ has the explicit form

$$\mathbf{T} = \begin{pmatrix} 1/t^* & (r/t) \\ (r/t)^* & 1/t \end{pmatrix} \tag{5}$$

where t and r are the amplitude transmission and reflection coefficients respectively.

The first step is to write for positive integer N

$$[t^*]^{-N} = [1,0]\otimes\ldots[1,0]\mathbf{T}\otimes\ldots\mathbf{T}\begin{pmatrix}1\\0\end{pmatrix}\otimes\ldots\begin{pmatrix}1\\0\end{pmatrix} \tag{6}$$

where each direct product is taken N times. Equation (4) generalises to

$$\mathbf{T}\otimes\ldots\mathbf{T} = \prod_{n=1}^{L} \mathbf{M}_n \otimes \ldots \mathbf{M}_n. \tag{7}$$

Equation (6) can be considerably simplified by noticing that any direct product of N identical matrices decomposes into independent subspaces according to the irreducible representations of the permutation group of order N [10]. For our purposes here it will be sufficient to notice that any symmetric vector, $\boldsymbol{u}$, that is a vector whose components have the property that

$$u_{j_1\ldots j_N} = u_{\mathrm{P}(j_1\ldots j_N)} \tag{8}$$

where P is any permutation of the N indices, remains symmetric after multiplication by (7). Since both projection vectors in (6) are symmetric their evolution under repeated multiplication is confined within the subspace of all symmetric vectors. To take advantage of this we must first choose a set of orthonormal basis vectors for the symmetric subspace. The set of vectors $\{\boldsymbol{v}(i) : i = 0, \ldots, N\}$ with components

$$v(i)_{j_1\ldots j_N} = \begin{cases} (i!(N-1)!/N!)^{1/2} & \text{if } j_1 + \ldots j_N = N + i \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

and each $j = 1$ or 2 is easily shown to be such a basis. We then define a symmetry-reduced transfer matrix $\mathbf{X}(\mathbf{T}, N)$ with elements

$$X_{i,j} = \boldsymbol{v}(i)\mathbf{T}\otimes\ldots\mathbf{T}\boldsymbol{v}(j) = \sum_{k=0}^{\min(i,j)} ({}^{N-i}C_{j-k}\,{}^{N-j}C_{i-k}\,{}^{i}C_k\,{}^{j}C_k)^{1/2} \times T_{1,1}{}^{N-i-j+k}\,T_{1,2}{}^{j-k}\,T_{2,1}{}^{i-k}\,T_{2,2}{}^{k} \tag{10}$$

It is easily shown that $\mathbf{X}$ has the usual multiplicative property of a transfer matrix

$$\mathbf{X}(\mathbf{T}, N) = \prod_{n=1}^{L} \mathbf{X}(\mathbf{M}_n, N). \tag{11}$$

Taking a further direct product allows us to construct the average

$$\langle |t|^{-2N} \rangle = [\boldsymbol{w}^{\mathrm{T}} \otimes \boldsymbol{w}^{\mathrm{T}}]\langle \mathbf{X}(\mathbf{M}, N) \otimes \mathbf{X}(\mathbf{M}^*, N)\rangle^{L}[\boldsymbol{w}\otimes\boldsymbol{w}]. \tag{12}$$

Here $\boldsymbol{w}$ is a column vector with components $w_j = \delta_{j,0}$, $\mathbf{M}^*$ is the complex conjugate of $\mathbf{M}$ and angular brackets indicate an average over the distribution $p(\mathbf{M})$.

To proceed further we need to continue (12) analytically in N. Kirkmann and Pendry [10] have shown in a previous paper that this can be achieved by generalising the definition of the $\mathbf{X}$ matrix. We analytically continue the binomial coefficients using the gamma function and allow the matrix indices to run over $0 \leqslant i, j \leqslant \infty$. This defines the $\mathbf{X}$ matrix for any N in the complex plane. We can see that the continuation is consistent with the previous exact result for the positive integer moments of $1/t$ by noticing that the matrix becomes block diagonal when N is a positive integer or zero

$$\mathbf{X}(N) = \left(\begin{array}{c|c} 0 \leqslant i, j \leqslant N & 0 \\ \hline 0 & N < i, j \leqslant \infty \end{array}\right). \tag{13}$$

However, for a detailed justification of the continuation procedure the reader is referred to [10].

We can now obtain the cumulants of the distribution of $-\ln|t|^2$ from (12) by successive differentiation

$$c_n(-\ln|t|^2) = (\mathrm{d}^n/\mathrm{d}N^n)\ln\langle|t|^{-2N}\rangle|_{N=0} \tag{14}$$

where c_n is the nth cumulant. In the limit that $L \to \infty$ a simple expression for the cumulants can be found subject to certain conditions on the eigenvalue spectrum of $\langle \mathbf{X} \otimes \mathbf{X}\rangle$. First we rewrite (12) in terms of the eigenvalues μ_m and eigenvectors of $\langle \mathbf{X} \otimes \mathbf{X}\rangle$ as

$$\langle|t|^{-2N}\rangle = \sum_{m=0}^{\infty} \mu_m(N)^L g(N, m) \tag{15}$$

with

$$g(N, m) = w^{\mathrm{T}} \otimes w^{\mathrm{T}} |N, m\rangle\langle m, N| w \otimes w. \tag{16}$$

Although the eigenvalue spectrum depends on the form of $p(\mathbf{M})$, some general points can be made. For $N < 0$ the strict bound $0 \leqslant \langle|t|^{-2N}\rangle \leqslant 1$ implies that $|\mu_m(N)| \leqslant 1$ for all m. At $N = 0$ $w \otimes w$ and $w^{\mathrm{T}} \otimes w^{\mathrm{T}}$ are eigenvectors of $\langle \mathbf{X} \otimes \mathbf{X}\rangle$ with eigenvalue unity which we label as $m = 0$. When $N > 0$ we find for the Anderson model, to be discussed in § 3, that $|\mu_0(N)| > 1$ and $|\mu_m(N)| < 1$ for $m > 0$. This will not necessarily be true for an arbitrary $p(\mathbf{M})$. Hence in the limit that $L \to \infty$ we have, subject to these conditions, the simple expression

$$c_n(-\ln|t|^2) = (\mathrm{d}^n/\mathrm{d}N^n)\,[L \ln \mu_0(N) + \ln g(N, 0)]|_{N=0}. \tag{17}$$

2.2. *The asymptotic distribution of* $-\ln|t|^2$

The most important feature of (17) is that all cumulants depend linearly on L in the long-length limit. This allows a simple form for the asymptotic distribution to be obtained. In the interest of brevity we set $z \equiv -\ln|t|^2$ so that

$$\langle|t|^{-2\mathrm{i}\alpha}\rangle = \int_{-\infty}^{+\infty} p(z)\exp(\mathrm{i}\alpha z)\,\mathrm{d}z. \tag{18}$$

Inverting the Fourier transform we find

$$p(z) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \exp[F_z(\alpha) - \alpha z]\,\mathrm{d}\alpha \tag{19}$$

where $F_z(\alpha)$ is the cumulant generating function defined as

$$F_z(\alpha) = \ln\langle\exp(\alpha z)\rangle. \tag{20}$$

In the limit that $L \to \infty$ this can be expressed as

$$p(z) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \exp[Lf_z(\alpha) - \alpha z]\,\mathrm{d}\alpha \tag{21}$$

where $F_z(\alpha) = Lf_z(\alpha)$ and f_z is independent of L. Evaluating the integral by the method of steepest descents we find

$$p(z) \to (2\pi c_2)^{-1/2}\exp[-(z - c_1)^2/2c_2]. \tag{22}$$

Thus $-\ln|t|^2$ is normally distributed in the limit that $L \to \infty$ with

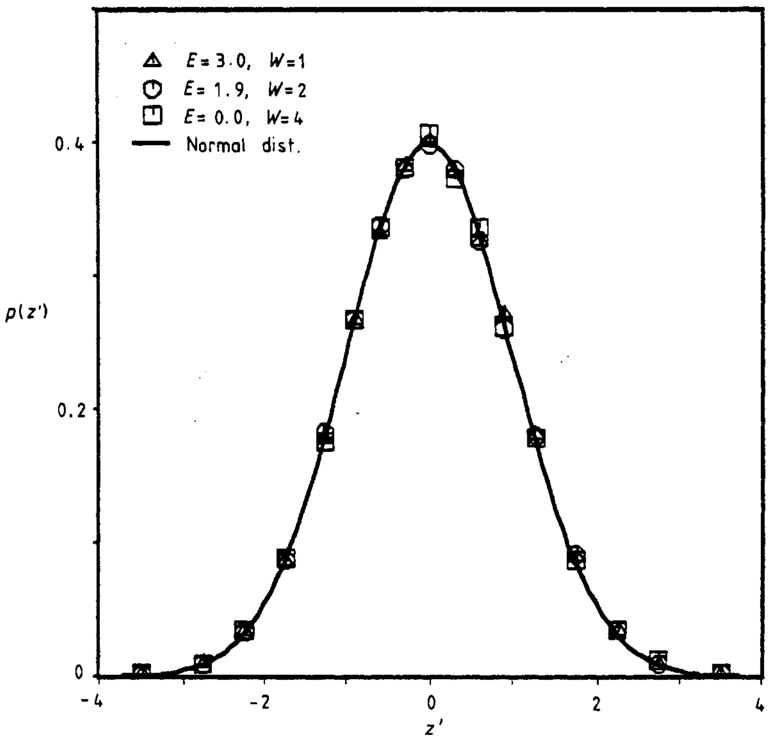

Figure 1. The distribution of the renormalised variable z' for various energies E and disorders W. The symbols are the results of a Monte Carlo simulation for systems of length $L = 1000$ obtained by sampling over 50 000 systems at random. The full curve line is the normal distribution expected on theoretical grounds.

$$c_1 = \langle z \rangle = L \frac{\partial}{\partial N} \ln \mu_0(N) \bigg|_{N=0} \qquad c_2 = \operatorname{var} z = L \frac{\mathrm{d}^2}{\mathrm{d}N^2} \ln \mu_0(N) \bigg|_{N=0}. \tag{23}$$

The distribution obeys a two-parameter scaling.

In figure 1 we plot the distribution of the renormalised variable z' where

$$z' = (z - \langle z \rangle)/(\operatorname{var} z)^{1/2} \tag{24}$$

is obtained from a Monte Carlo simulation of the Anderson model. This renormalisation allows different data to be displayed on the same graph. We see from this figure that in all cases z' is normally distributed as required by (22).

2.3. *The accuracy of the asymptotic distribution*

In this section we discuss for what values of z and L the asymptotic form (22) is accurate.

The range of z implied by (22) is $[-\infty, +\infty]$ whereas it is clear that in reality $|t| \leq 1$ so that $z \geq 0$. This contradiction is resolved by noting that in the limit $L \to \infty$ the integrated density over the unphysical region $[-\infty, 0]$ is zero. At finite lengths the integrated density will be almost zero if $L \gg L_c$, where L_c is the localisation length defined as

$$L_c^{-1} = \lim_{L\to\infty} \langle z\rangle/2L. \tag{25}$$

It is important to realise that even if $L \gg L_c$ the asymptotic form (22) may not give the moments of $|t|^2$ correctly. Let us consider the nth moment where n may be positive or negative; by definition

$$\langle |t|^{2n}\rangle = \int_{-\infty}^{+\infty} p(z)\exp(-nz)\,\mathrm{d}z. \tag{26}$$

The integrand has a maximum in the unphysical region $z < 0$ if $n \geq n_0$ where

$$n_0 = \langle z\rangle/\mathrm{var}\, z. \tag{27}$$

For the weak-disorder limit of the Anderson model $n_0 = \frac{1}{2}$. In fact for large enough positive n the moments approximated using the asymptotic form (22) exceed unity. We conclude that the positive moments of $|t|^2$ are never given correctly by the asymptotic distribution.

This leaves open the possibility that the negative moments may be given accurately by (22). To determine that this is in fact the case we examine the saddle point approximation more carefully. The position of the saddle point α_0 is determined by the condition.

$$(\mathrm{d}/\mathrm{d}\alpha)\,[Lf_z(\alpha) - \alpha z]|_{\alpha=\alpha_0} = 0. \tag{28}$$

To find the root of this equation it is usual to truncate $f_z(\alpha)$ at order α^2 giving

$$\alpha_0 = (z - c_1)/c_2. \tag{29}$$

This truncation is valid provided $\alpha_0 \ll 1$.

The condition that $\alpha_0 \ll 1$ means that (22) is a bad approximation to $p(z)$ for values of z far from the mean, that is for values of z in the tails of the distribution. Since the negative moments are sensitive to the positive tail of $p(z)$, as is clear from the form of the integrand (26) when $n < 0$, we find that the negative moments will in general be inconsistent with the normal approximation for $p(z)$. The exception to this occurs when the higher cumulants of z are zero and $f_z(\alpha)$ actually terminates at order α^2. Then the restriction to $\alpha_0 \ll 1$ no longer applies and the positive tail is accurately Gaussian. In this case the negative moments of $|t|^2$ are consistent with the normal distribution for z.

2.4. The phase distribution

The cumulants of the phase distribution are evaluated in a similar manner to those for z starting from the expression

$$\langle \exp(N\Theta)\rangle = [w^{\mathrm{T}} \otimes w^{\mathrm{T}}]\langle \mathbf{X}(\mathbf{M}, -\mathrm{i}N/2) \otimes \mathbf{X}(\mathbf{M}^*, \mathrm{i}N/2)\rangle^L [w \otimes w] \tag{30}$$

In an exactly analogous analysis to § 2.2 we find

$$p(\Theta) \to \frac{1}{(2\pi\,\mathrm{var}\,\Theta)^{1/2}} \exp\left(\frac{-(\Theta - \langle\Theta\rangle)^2}{2\,\mathrm{var}\,\Theta}\right) \tag{31}$$

with

$$\langle\Theta\rangle = L(\mathrm{d}/\mathrm{d}N)\ln\mu_0(N)|_{N=0} \tag{32}$$

and

$$\mathrm{var}\,\Theta = L(\mathrm{d}^2/\mathrm{d}N^2)\ln\mu_0(N)|_{N=0} \tag{33}$$

where $\mu_0(N)$ is the $m = 0$ eigenvalue of the operator

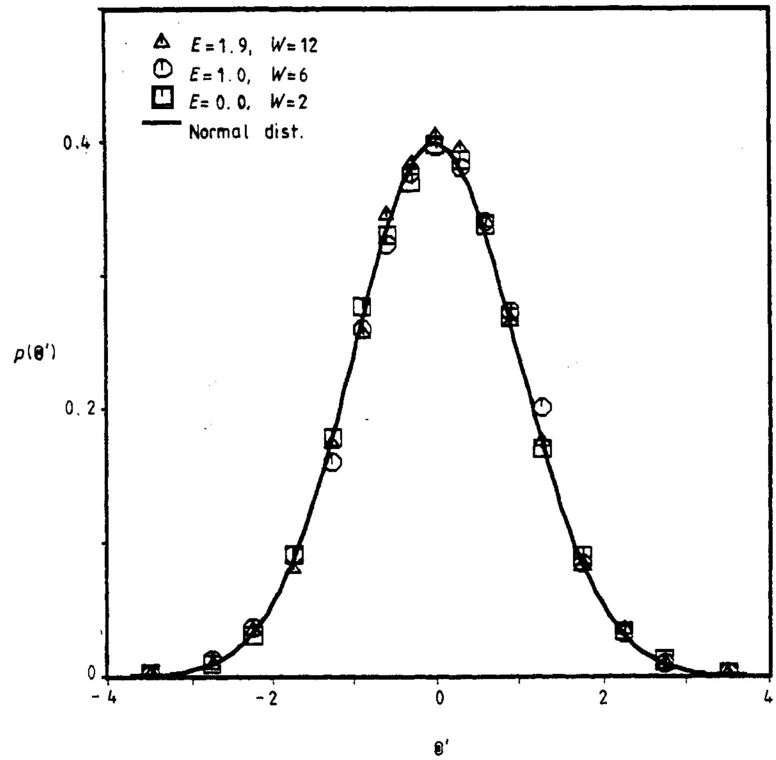

Figure 2. The distribution of the renormalised phase Θ' for various energies E and disorders W. The symbols are the results of a Monte Carlo simulation, the full curve is the normal distribution expected on theoretical grounds.

$$\langle \mathbf{X}(\mathbf{M}, -\mathrm{i}N/2) \otimes \mathbf{X}(\mathbf{M}^*, \mathrm{i}N/2)\rangle. \tag{34}$$

In figure 2 we plot the distribution of the renormalised variable Θ'

$$\Theta' = (\Theta - \langle\Theta\rangle)/(\operatorname{var}\Theta)^{1/2} \tag{35}$$

obtained from a Monte Carlo simulation of the 1D Anderson model with diagonal disorder. Again the renormalised results fit very closely to a normal distribution with mean zero and variance unity, as required by (31).

Note that since Θ is normally, as opposed to log–normally, distributed we can say immediately that the higher moments of Θ are consistent with (31) only if the higher cumulants are zero.

It is worth inserting a word of caution at this point. Like other quantities reviewed in this paper the phase distribution contains a wealth of subtleties in the fine structure. In particular the contribution of resonant states within the disordered region gives rise to structure in both the phase of reflection and transmission (they are closely related by time-reversal arguments) which is important in the theory of $1/f$ noise at surfaces. We shall not discuss these effects here but refer the reader to earlier papers: see [14–16].

3. Application to the Anderson model

3.1. The weak-disorder limit of the Anderson model

The Hamiltonian **H** for the Anderson model with diagonal disorder is

$$Ea_n = \varepsilon_n a_n - a_{n+1} - a_{n-1}. \tag{36}$$

We suppose the site energies $\{\varepsilon_n : n = 1, L\}$ to be independently and identically distributed according to a distribution $p(\varepsilon)$ with $\langle\varepsilon\rangle = 0$. The transfer matrix for the model is given by (4) with

$$\mathbf{M}_n = \begin{pmatrix} (1 - \mathrm{i}\delta_n)\mathrm{e}^{\mathrm{i}k} & -\mathrm{i}\delta_n \mathrm{e}^{\mathrm{i}k} \\ \mathrm{i}\delta_n \mathrm{e}^{\mathrm{i}k} & (1 + \mathrm{i}\delta_n)\mathrm{e}^{-\mathrm{i}k} \end{pmatrix} \tag{37}$$

where the wavenumber k is related to the energy E by

$$E - \langle\varepsilon\rangle = -2\cos k \tag{38}$$

and

$$\delta_n = (\varepsilon_n - \langle\varepsilon\rangle)/2\sin k. \tag{39}$$

In the absence of disorder for $|E| < 2$, that is inside the band of the pure system, we have

$$[\langle \mathbf{X}(\mathbf{M}, N) \otimes \mathbf{X}(\mathbf{M}^*, N)\rangle]_{mm',nn'} = \delta_{m,n}\delta_{m,n'}\, \mathrm{e}^{2\mathrm{i}k(m'-m)}. \tag{40}$$

The zero-order eigenvalues are highly degenerate and in particular there is a large degeneracy of the eigenvalues at unity ($m = m'$). The introduction of disorder breaks this degeneracy. The new eigenvalues can be estimated from degenerate perturbation theory, and working to first order in var ε we obtain a tridiagonal perturbation matrix $\mathbf{\Omega}$

$$\mathbf{\Omega}(N) = \mathbf{I} + \frac{\operatorname{var}\varepsilon}{E^2 - 4}\mathbf{\Omega}'(N) \tag{41}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{\Omega}'$ has elements $(0 \leqslant j \leqslant \infty)$

$$\Omega'_{j,j} = N(1 + 2j) - 2j^2 \tag{42}$$
$$\Omega'_{j,j+1} = (N - j)(j + 1).$$

Here we neglect extra degeneracy which occurs at rational k-values and which modifies the results only at $E = 0$. We return to this point later.

The perturbation matrix is real symmetric and therefore its eigenvalues are real. At $N = 0$ we find as required the $\mu_0 = 1$ and that

$$\mu_m(N = 0) < 1 - \frac{\operatorname{var}\varepsilon}{4(E^2 - 4)} \qquad m > 0. \tag{43}$$

Using the weak-disorder result for the localisation length [8] this gives

$$(\mu_m/\mu_0)^L < \exp(-L/2L_c) \tag{44}$$

so that the central result for the cumulants (17) is valid provided that $L \gg L_c$.

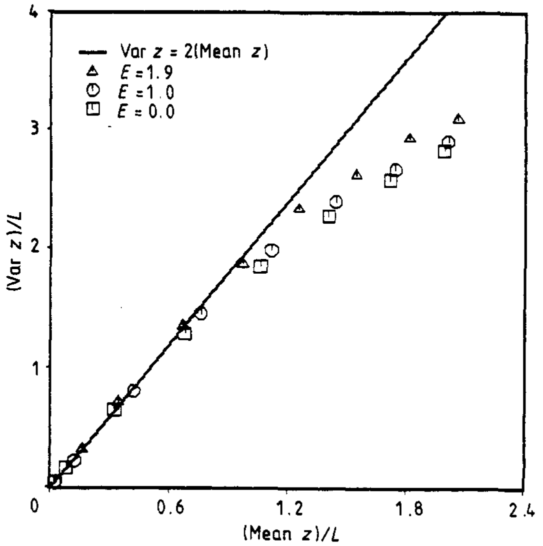

Figure 3. The relationship between the mean and variance of z for various energies E and disorders W ($W = 1, 2, 4, 5, 8, 10, 12, 14$). The figure shows the breakdown of single-parameter scaling for strong disorder. These results are for energies within the band of the pure system.

Using (17) and expanding the logarithm to first order in var ε we find

$$c_n(-\ln|t|^2) = L\frac{\operatorname{var}\varepsilon}{E^2 - 4}\frac{\mathrm{d}^n}{\mathrm{d}N^n}\mu_0'(N)|_{N=0} \tag{45}$$

where $\mu_0'(N)$ is the relevant eigenvalue of $\boldsymbol{\Omega}'$. The first derivative of μ_0' at $N = 0$ is easily shown to be unity. The higher derivatives are obtained numerically by truncating $\boldsymbol{\Omega}'$ at various sizes and converging the derivatives against truncation size. Using this procedure we find

$$\langle -\ln|t|^2\rangle = L\frac{\operatorname{var}\varepsilon}{4 - E^2} \qquad \operatorname{var}(-\ln|t|^2) = 2L\frac{\operatorname{var}\varepsilon}{4 - E^2} \tag{46}$$

giving a direct relation between the mean and variance:

$$\operatorname{var}(-\ln|t|^2) = 2\langle -\ln|t|^2\rangle. \tag{47}$$

This means that the asymptotic distribution of $-\ln|t|^2$ is determined by a single parameter, $\langle -\ln|t|^2\rangle$, in the weak-disorder limit. In figure 3 we test the range of validity of the single-parameter scaling method for the Anderson model with

$$p(\varepsilon) = 1/W \qquad |\varepsilon| \leqslant \tfrac{1}{2}W \tag{48}$$

by means of a Monte-Carlo simulation. We find deviations from (47) for disorders W of the order of twice the band width. This does not in itself imply a crossover to two-parameter scaling as it would be sufficient for a single-parameter scaling if all the results in figure 3 fell on a common curve. However, it is clear this is not the case and that when Lc is of the order of a few lattice sites there is a weak dependence on a second parameter.

At $E = 0$ there is a well-known modification [10, 17] to the result for the mean

$$\langle -\ln|t|^2\rangle = 0.4569\, l\frac{\operatorname{var}\varepsilon}{4 - E^2} \tag{49}$$

a result which can be obtained by considering $\langle \mathbf{X}\rangle$ alone [10] without the complication

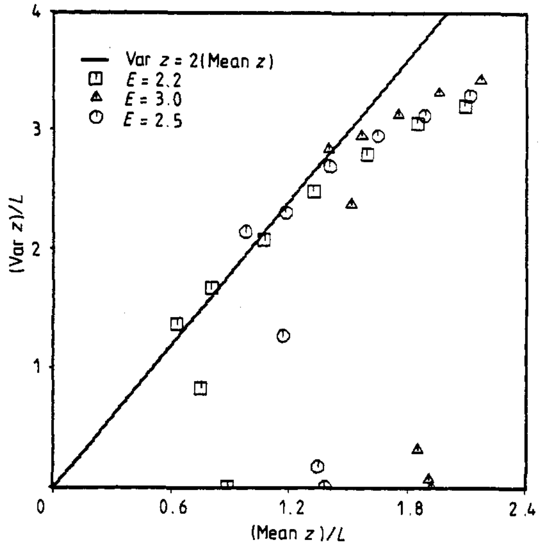

Figure 4. The relationship between the mean and variance of z for energies outside the band of the pure system and disorders W ($W = 1$–14). Here single-parameter scaling is not valid even in the weak-disorder limit.

of a further direct product. The Monte-Carlo simulation indicates that the relationship between the mean and variance (47) still holds at $E = 0$.

When $|E| > 2$, that is outside the energy band of the pure system, we have

$$[\langle \mathbf{X}(\mathbf{M}, N) \otimes \mathbf{X}(\mathbf{M}^*, N)\rangle]_{mm',nn'} = \delta_{m,n}\delta_{m',n'} \exp[2ik(N - m' - m] \qquad (50)$$

since ik is now real. At $N = 0$ the $m = 0$ eigenvalue is no longer degenerate and we find using non-degenerate perturbation theory that

$$\langle -1|t|^2\rangle = L\left(2ik - \frac{\text{var}\,\varepsilon}{E^2 - 4}\right) \qquad \text{var}(-\ln|t|^2) = 4\frac{\text{var}\,\varepsilon}{E^2 - 4}. \qquad (51)$$

The absence of a direct relationship between the mean and variance implies a two-parameter scaling even for weak disorder, a result confirmed by the Monte Carlo results presented in figure 4.

The analysis for the phase distribution follows similar lines. For $|E| < 2$ we find

$$\langle \Theta \rangle = kL \qquad \text{var}\,\Theta = \tfrac{3}{2}L\frac{\text{var}\,\varepsilon}{4 - E^2}. \qquad (52)$$

This implies a two-parameter scaling of the Θ-distribution. When $|E| > 2$ we find

$$\langle \Theta \rangle = \begin{cases} 0 & E < -2 \\ \pi & E > 2 \end{cases}$$
$$\text{var}\,\Theta = 0. \qquad (53)$$

Here the fluctuations in Θ are quenched as the density of states tends to zero beyond $E = \pm(2 + W/2)$ in accordance with the Saxon–Hutner conjecture, [18]. In figure 5 these results are confirmed in a Monte Carlo simulation of the Anderson model.

3.2. *Weak-disorder results for the higher cumulants*

For $|E| < 2$ we find that $c_3(-\ln|t|^2)$ and $c_4(-\ln|t|^2)$ are both zero. For the higher cumulants we are not able to obtain convergence of the necessary derivatives against

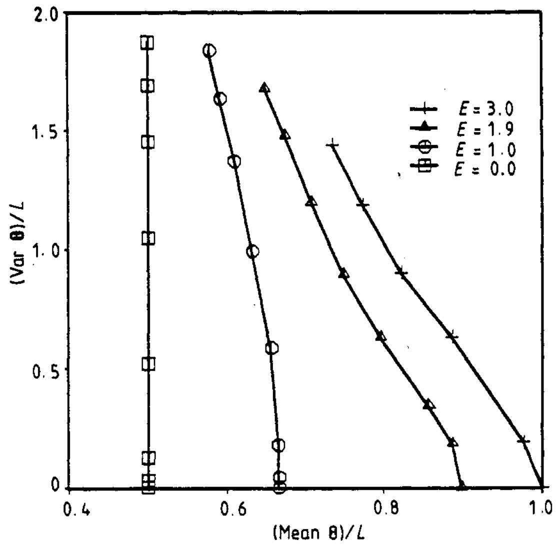

Figure 5. The relationship between the mean and variance of Θ' for various energies and disorders ($W = 1$–12), showing two-parameter scaling of the distribution.

the truncation size of the perturbation matrix $\boldsymbol{\Omega}'$. Nevertheless, we do find that the higher derivatives decrease with increasing truncation size suggesting that they are in fact zero. When $|E| > 2$ we find from non-degenerate perturbation theory that all higher cumulants are zero. This leads us to conjecture that to first order in the statistical variance

$$c_n(-\ln|t|^2) = 0 \qquad n > 2. \tag{54}$$

With reference to the discussion of section 2 this implies that the negative moments of $|t|^2$ will be consistent with the normal distribution of $-\ln|t|^2$ in the weak-disorder limit.

The higher cumulants of the phase distribution also appear to be zero, indicating that all the moments of Θ agree with (31) in the weak-disorder limit.

3.3. Strong disorder

Assuming that the form of the eigenvalue spectra is not radically altered for strong disorder, both $-\ln|t|^2$ and Θ should be normally distributed as $L \to \infty$ for arbitrary disorder strength W. However, the result of section 3.2 that all higher cumulants are zero to first order in var ε will probably be modified at higher orders. For strong disorder we therefore expect that the negative, in addition to the positive moments, of $|t|^2$ and Θ will be inconsistent with the asymptotic distribution. This has been shown explicitly for the negative moments in the very strong-disorder limit by a direct calculation [19].

4. Conclusions

In conclusion we have found that in the long-length limit $p(-\ln|t|^2)$ approaches a normal distribution, obeying in general a two-parameter scaling, but that the normal approximation is never accurate enough to allow calculation of the positive moments of

$|t|^2$, nor the negative moments except for weak disorder. In fact the positive moments are of the form [20]

$$\langle |t|^{2N} \rangle = A(N)[\alpha_0/(\alpha_2 L)^{3/2}] \exp(\alpha_1 L) \tag{55}$$

quite at variance with the normal distribution for $-\ln|t|^2$. Here $A(N)$ is a known numerical factor independent of the disorder and the α_i are parameters which depend on the disorder but which cannot in general be found analytically.

The results of this paper concerning the statistical properties of $|t|^2$ can be related to those of the conductance g and the resistance R through the relations

$$g = R^{-1} = |t|^2. \tag{56}$$

This implies that $p(R)$ is accurately log–normal, in the sense that the moments of R are consistent with the log–normal distribution, for weak disorder only. The conductance distribution $p(g)$ is never accurately log–normal.

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